Actuator Placement in Structural Control

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The placement of force and torque actuators for structural control problems is considered. Objective functions are defined based on the elements of the actuator influence matrix, and optimization studies are conducted. The performance of the control is compared. The results indicate that a relatively even distribution of the actuators gives satisfactory results, whereas a close spacing of the actuators leads to excessive fuel and energy use. In all cases, several evenly spaced actuator distributions are found to be suitable. In addition, torque actuators are found to be less desirable than force actuators because they excite the higher modes to a greater degree. The efficiency of piecewise-continuous actuators is analyzed. Piecewise-continuous actuators are more realistic models than point actuators and they reduce stress levels. However, if their contact area is too large, they use more fuel.

I. Introduction

An important issue in the control design for elastic systems is the determination of the optimal number and location of the control system components: actuators and sensors, as well as their backups. In many cases, the efficiency and performance of a control law is affected by the placement of sensors and actuators.

Recent work in determining the number and location of control system components in distributed-parameter systems includes Refs. 1-14, where a variety of criteria have been considered. References 1-3 use an optimal control cost minimization criterion. Minimization of the control energy is considered in Ref. 4, and minimization of the expected value of a performance index is sought in Ref. 5. Reliability issues are utilized in Refs. 6 and 7. In Ref. 6, mission life is considered, and in Ref. 7, possible actuator failures and controllability after failure are considered. The degree of controllability is also used as a criterion in Ref. 8. Failures of actuators and sensors are investigated in Refs. 9 and 10, respectively, where criteria are proposed to locate the actuators and sensors in a way to facilitate the failure detection process. The results of Refs. 4, 9, and 10 indicate that an even distribution of the control components generally yields satisfactory results. General reliability and replacement of control components is considered in Refs. 11 and 12. In Ref. 13, the accuracy with which the modal coordinates are extracted from the sensors output is used as a criterion to place the sensors. Minimization of power is used as a criterion in Ref. 14 to locate both the sensors and actuators.

In this paper, we consider two issues associated with placement of actuators on a structure: what types of actuators to use and where on the structure to locate the given actuators. We analyze point as well as piecewise-continuous actuators and conduct optimization studies using objective functions based on controllability of a certain mode and coupling among the modes. We then compare the performance of the control for the various actuator locations, for fuel use, and for work done. Intuitively, and considering our sensor placement results,[13] one expects that a relatively even distribution of the actuators should lead to satisfactory results. The optimization studies lead to a similar conclusion. Furthermore, it is found that for every objective function there are several local minima, so that there are many distributions of the actuators that are suitable.

II. Equations of Motion

Consider a flexible structure whose evolution is described in the form

$$\mathcal{L}u(x,t) + m(x)\ddot{u}(x,t) = f(x,t), \qquad 0<x<L \tag{1}$$

where $u(x,t)$ is the deformation at x, $m(x)$ the mass distribution, $\mathcal{L}$ a linear differential stiffness operator, and $f(x,t)$ the external excitation. The associated eigensolution consists a set of eigenvalues λ_r, and associated eigenfunctions $\phi_r(x)$. The modal equations have the form

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \qquad r = 1, 2, \ldots \tag{2}$$

in which $u_r(t)$ are modal coordinates and $f_r(t)$ are modal forces related to $u(x,t)$ and $f(x,t)$ by

$$u(x,t) = \sum_{r=1}^{\infty} \phi_r(x)u_r(t), \qquad r = 1, 2, \ldots \tag{3a}$$

$$u_r(t) = [u(x,t),\ m(x)\phi_r(x)], \qquad r = 1, 2, \ldots \tag{3b}$$

$$f(x,t) = \sum_{r=1}^{\infty} m(x)\phi_r(x)f_r(t), \qquad r = 1, 2, \ldots \tag{3c}$$

$$f_r(t) = [f(x,t),\ \phi_r(x)], \qquad r = 1, 2, \ldots \tag{3d}$$

where $[a,b] = \int ab\,\mathrm{d}x$.

Spatially continuously distributed measurements and controls are not within the state of the art. We will consider four types of discrete actuators: point force actuators, piecewise-continuous force actuators, point torquers, and piecewise-continuous torquers. For point force actuators, we express the force profile as

$$f(x,t) = \sum_{i=1}^{k} F_i(t)\delta(x - x_{ai}) \tag{4}$$

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in which x_{ai} are the actuator locations and k is the number of actuators. The modal forces become

$$f_r(t) = \sum_{i=1}^{k} \int_0^L F_i(t)\delta(x - x_{ai})\phi_r(x)\,\mathrm{d}x$$

$$= \sum_{i=1}^{k} B_{ri}F_i(t), \qquad r = 1, 2, \ldots \tag{5}$$

where $B_{ri} = \phi_r(x_{ai})$ $(r = 1, 2, \ldots;\ i = 1, 2, \ldots, k)$ are the entries of the actuator influence matrix B. The modal equations of motion can be expressed in matrix form

$$\ddot{u}(t) + \Lambda u(t) = f(t) = BF(t) \tag{6}$$

where $u(t)$ and $f(t)$ are infinite-dimensional vectors containing the modal coordinates and modal forces, Λ is a diagonal matrix containing the eigenvalues, and $F(t)$ contains the actuator inputs.

Next, consider control forces that are piecewise continuous. The contact length for each actuator is $2\epsilon_i$, with the midpoint located at $x_{ai}(i = 1, 2, \ldots, k)$. Each actuator imparts a total force of $F_i(t)$ $(i = 1, 2, \ldots, k)$. The force distribution can be expressed as[15]

$$f(x,t) = \sum_{i=1}^{k} \frac{F_i(t)}{2\epsilon_i}\,[u(x - x_{ai} + \epsilon_i) - u(x - x_{ai} - \epsilon_i)] \tag{7}$$

where u is the unit step function. We observe that $f(x,t)$ is now a piecewise-continuous function and that $F_i(t)/2\epsilon_i$ denotes the force density of the ith actuator. It follows that

$$f_r(t) = \sum_{i=1}^{k} \int_0^L \frac{F_i(t)}{2\epsilon_i}\,[u(x - x_{ai} + \epsilon_i) - u(x - x_{ai} - \epsilon_i)]$$

$$\times \phi_r(x)\,\mathrm{d}x = \sum_{i=1}^{k} B_{ri}F_i(t), \qquad r = 1, 2, \ldots \tag{8}$$

where

$$B_{ri} = \frac{1}{2\epsilon_i}\int_{x_{ai}-\epsilon_i}^{x_{ai}+\epsilon_i} \phi_r(x)\,\mathrm{d}x \tag{9}$$

For point torquers located at $x_{ti}(i = 1, 2, \ldots, k)$, one can express the external excitation as[15]

$$f(x,t) = -\partial \bar{m}(x,t)/\partial x = \sum_{i=1}^{k} -M_i(t)\delta'(x - x_{ti}) \tag{10}$$

in which $\bar{m}(x,t)$ is the spatial torque distribution, $M_i(t)$ denote the amplitudes of the torque inputs of actuators located at $x_{ti}(i = 1, 2, \ldots, k)$, and δ' is the spatial derivative of the Dirac Delta function.[16] Substitution of Eq. (10) into the expansion theorem [Eq. (3d)] yields

$$f_r(t) = \sum_{i=1}^{k} \int_0^L -M_i(t)\delta'(x - x_{ti})$$

$$\times \phi_r(x)\,\mathrm{d}x = \sum_{i=1}^{k} B_{ri}M_i(t), \qquad r = 1, 2, \ldots \tag{11}$$

where

$$B_{ri} = \int_0^L -\delta'(x - x_{ti})\phi_r(x)\,\mathrm{d}x = \phi_r'(x_{ti})$$

$$r = 1, 2, \ldots, n; \qquad i = 1, 2, \ldots, k \tag{12}$$

In a similar fashion, for piecewise-continuous torquers denoting by $M_i(t)$ the total torque, the moment distribution can be expressed as

$$\bar{m}(x,t) = \sum_{i=1}^{k} \frac{M_i(t)}{2\epsilon_i}\,[u(x - x_{ti} + \epsilon_i) - u(x - x_{ti} - \epsilon_i)] \tag{13}$$

Substituting Eq. (13) into the expansion theorem, we obtain for the entries of the actuator influence matrix

$$B_{ri} = \int_0^L -\frac{1}{2\epsilon_i}\,[\delta'(x - x_{ti} + \epsilon_i) - \delta'(x - x_{ti} - \epsilon_i)]\phi_r(x)\,\mathrm{d}x$$

$$= \frac{1}{2\epsilon_i}\,[\phi_r'(x - x_{ti} + \epsilon_i) - \phi_r'(x - x_{ti} - \epsilon_i)] \tag{14}$$

It is shown in Ref. 15 that the elements of the actuator influence matrix B are of the largest order for point torquers and smallest for piecewise-continuous force actuators. For example, for point force actuators, the entries of the actuator influence matrix B_{ri} are of order $\mathcal{O}[\phi_r(x)]$ $(r = 1, 2, \ldots)$. For point torque actuators, $B_{ri} = \mathcal{O}[\phi_r'(x)] = \mathcal{O}[r\phi_r(x)]$, $(r = 1,2,\ldots)$. This implies that torque actuators excite the elastic motion more than force actuators, especially for the higher modes. Also, piecewise-continuous actuators excite the elastic motion to a lesser degree than point actuators.

III. Performance Measures

We now define performance measures associated with the control effort. The first measure is the work done by the actuators on the structure and is defined for distributed force actuators as

$$W = \int_0^T\int_0^L f(x,t)\dot{u}(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{15}$$

where T is the final time. We will refer to this expression as the physical work. We next define the actual work (or absolute work) as

$$W_A = \int_0^T\int_0^L |f(x,t)\dot{u}(x,t)|\,\mathrm{d}x\,\mathrm{d}t \tag{16}$$

where the absolute value signs are introduced to account for both positive and negative work. This expression indicates the actual total work done by the actuators, whereas W gives the difference between the initial and final energy levels. Ideally, for vibration suppression, one would like to design a control law that does only negative work. However, depending on the control method, the sign of the velocities at the actuators' locations can cause positive work to be done. Positive work in vibration suppression implies wasted energy and excitation of undesirable motion.

For point forces, the actual work becomes[15]

$$W_A = \sum_{j=1}^{k}\int_0^T |F_j(t)\dot{u}(x_{aj},t)|\,\mathrm{d}t \tag{17}$$

and for piecewise-continuous actuators

$$W_A = \sum_{i=1}^{k}\int_0^T\int_{x_{ai}-\epsilon_i}^{x_{ai}+\epsilon_i} \left|\frac{F_i(t)}{2\epsilon_i}\,\dot{u}(x,t)\right|\,\mathrm{d}x\,\mathrm{d}t \tag{18}$$

which is based on the assumption that the entire length of the actuator is in contact with the beam at all times. Similarly, for point torquers and piecewise-continuous torquers, we have, respectively,

$$W_A = \sum_{i=1}^{k}\int_0^T |M_i(t)\dot{u}'(x_{ti},t)|\,\mathrm{d}t \tag{19a}$$

$$W_A = \sum_{i=1}^{k}\int_0^T\int_{x_{ti}-\epsilon_i}^{x_{ti}+\epsilon_i} \left|\frac{M_i(t)}{2\epsilon_i}\,\dot{u}'(x,t)\right|\,\mathrm{d}x\,\mathrm{d}t \tag{19b}$$

Next, we define modal work, denoted by W_r, by describing the work done by the modal forces on the modal coordinates. This expression is not a physical quantity. However, it gives a measure of how each mode is controlled and whether there is energy transfer from one mode to another. Energy transferred from one mode to another increases the time it takes for the control to take effect and it also increases fuel consumption. For the rth mode, the modal work has the form

$$W_r = \int_0^T |f_r(t)\dot{u}_r(t)| \, dt, \qquad r = 1, 2, \ldots \tag{20}$$

where, again, absolute value signs are used to add the contributions of the positive and negative work. The total modal work is denoted by W_M and defined by $W_M = \sum_{r=1}^{\infty} W_r$. Note that for the general case of control the total modal work is not necessarily equal to the actual work, $W_A \neq W_M$.

The modal energy, or Hamiltonian, for each mode is defined as $H_r = [\dot{u}_r^2 + \omega_r^2 u_r^2]/2$ $(r = 1, 2, \ldots)$. The total modal energy H, which is the same as the physical energy, is

$$H = \sum_{r=1}^{\infty} H_r$$

The fuel, or impulse, is defined for discrete force actuators as

$$\mathfrak{F} = \int_0^T \int_0^L |f(x,t)| \, dx \, dt = \sum_{j=1}^{m} \int_0^T |F_j(t)| \, dt \tag{21}$$

For discrete torques, the fuel, which becomes a measure of the angular impulse, is defined as

$$\mathfrak{F} = \int_0^T \int_0^L |m(x,t)| \, dx \, dt = \sum_{j=1}^{m} \int_0^T |M_j(t)| \, dt \tag{22}$$

We introduce here a new performance measure, namely, the work-to-fuel ratio, denoted by $W/\mathfrak{F}$. This ratio is a measure of the effectiveness of the control action and it compares the amount of fuel required to do a certain amount of work. A higher value of this ratio indicates a more effective control. Note that for force actuators the dimension of the work-to-fuel ratio is the unit of speed.

IV. Control Laws

We will evaluate the performance of the actuator placement criteria using centralized and decentralized control laws. A common type of centralized control is modal control, where a set of modes are targeted for monitoring and control. Partitioning the modal coordinates as $\boldsymbol{u}(t) = [\boldsymbol{u}_M^T(t) | \boldsymbol{u}_R^T(t)]^T$, one can write the modal equations in the form

$$\ddot{\boldsymbol{u}}_M(t) + \Lambda_M \boldsymbol{u}_M(t) = \boldsymbol{f}_M(t) = B_M \boldsymbol{F}(t) \tag{23a}$$

$$\ddot{\boldsymbol{u}}_R(t) + \Lambda_R \boldsymbol{u}_R(t) = \boldsymbol{f}_R(t) = B_R \boldsymbol{F}(t) \tag{23b}$$

with terms taking their obvious meaning. For linear control,

$$\boldsymbol{F}_A(t) = K_1(t)\boldsymbol{u}_M(t) + K_2(t)\dot{\boldsymbol{u}}_M(t) \tag{24}$$

in which $K_1(t)$ and $K_2(t)$ are gain matrices. If they are designed such that $B_M K_1$ and $B_M K_2$ are not diagonal, then there is energy transfer from one mode to another, a phenomenon described as control spillback.[17] We also observe that the work done by the controller is not always negative.

One can show that for spatially continuously distributed actuators and when only velocity feedback is used $W_A = W_M$. First, the modal forces are designed independently as[18]

$$f_r(t) = \alpha_r \dot{u}_r(t), \qquad r = 1, 2, \ldots \tag{25}$$

where α_r is a proportionality constant. Introducing the modal force into Eq. (20) we obtain

$$W_r = \int_0^T |f_r(x)\dot{u}_r(t)| \, dt = \int_0^T |\alpha_r [\dot{u}_r(t)]^2| \, dt = \int_0^T \alpha_r [\dot{u}_r(t)]^2 \, dt \tag{26}$$

Introducing Eqs. (3c) and (25) into Eq. (16), we obtain

$$\begin{aligned} W_A &= \int_0^T \int_0^L |f(x,t)\dot{u}(x,t)| \, dx \, dt \\ &= \int_0^T \int_0^L \left| \sum_{s=1}^{\infty} m(x)\phi_s(x)\alpha_s \dot{u}_s(t) \sum_{r=1}^{\infty} \phi_r(x)\dot{u}_r(t) \right| dx \, dt \\ &= \int_0^T \int_0^L \left| \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} m(x)\phi_r(x)\phi_s(x)\alpha_s \dot{u}_s(t)\dot{u}_r(t) \right| dx \, dt \\ &= \int_0^T \left| \sum_{r=1}^{\infty} \alpha_r [\dot{u}_r(t)]^2 \right| dt = \int_0^T \sum_{r=1}^{\infty} \alpha_r [\dot{u}_r(t)]^2 \, dt \\ &= \sum_{r=1}^{\infty} W_r = W_M \end{aligned} \tag{27}$$

In decentralized collocated control, the force vector has the form (for force actuators)

$$\boldsymbol{F}(t) = K_1'(t)\boldsymbol{y}(t) + K_2'(t)\dot{\boldsymbol{y}}(t) \tag{28}$$

in which $\boldsymbol{y}(t)$ denotes the system output at the actuator locations as $\boldsymbol{y}(t) = [u(x_{a1},t)u(x_{a2},t) \cdots u(x_{ak},t)]^T$ and K_1' and K_2' are diagonal gain matrices. For torque actuators, we replace $\boldsymbol{y}(t)$ and $\dot{\boldsymbol{y}}(t)$ in Eq. (28) by their spatial derivatives. The actuator locations can be expressed in terms of the modal coordinates as $\boldsymbol{y}(t) = B^T\boldsymbol{u}(t)$, leading to the closed-loop equation

$$\ddot{\boldsymbol{u}}(t) + \Lambda \boldsymbol{u}(t) = BK_1'(t)B^T\boldsymbol{u}(t) + BK_2'(t)B^T\dot{\boldsymbol{u}}(t) + \boldsymbol{n}(t) \tag{29}$$

For velocity feedback alone, decentralized control leads to a system whose physical energy continuously decays. From Eq. (28), we can write

$$F_j(t) = K_j \dot{u}(x_{aj},t), \qquad j = 1, 2, \ldots, k \tag{30}$$

which, when substituted into the actual work expression in Eq. (16), yields

$$W_A = \sum_{j=1}^{k} \int_0^T |F_j(t)\dot{u}(x_{aj},t)| \, dt = \sum_{j=1}^{k} \int_0^T K_j(t)[\dot{u}(x_{aj},t)]^2 \, dt \tag{31}$$

indicating the continuous decay in the system energy.

Although in decentralized control the physical energy is always decaying, there is energy transfer from one mode to the other. To demonstrate this, we introduce the expansion

$$\dot{u}(x_{aj},t) = \sum_{r=1}^{\infty} \phi_r(x_{aj})\dot{u}_r(t)$$

to Eq. (31), which yields

$$\begin{aligned} W_A &= \sum_{j=1}^{k} \int_0^T K_j(t) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \phi_r(x_{aj})\phi_s(x_{aj})\dot{u}_r(t)\dot{u}_s(t) \, dt \\ &= \int_0^T \dot{\boldsymbol{u}}^T(t)BKB^T\dot{\boldsymbol{u}}(t) \, dt \neq \sum_{j=1}^{\infty} W_r \end{aligned} \tag{32}$$

The inequality is present because of the off-diagonal entries of BKB^T. It follows that for modal control laws the physical work may be wasted because positive work can be carried out, whereas for decentralized laws, there is energy transfer from one mode to another. The question arises whether one can minimize both the modal and physical work.

V. Objective Functions

We consider a variety of objective functions to determine the optimal locations of the actuators. The objective functions are based on the entries of the actuator influence matrix B.

They are independent of time, thus giving general measures of controllability. We assume that the mathematical model consists of n modes.

The first objective function, denoted by J_1, is selected to measure the general controllability of modal control without considering a particular control method and considering that the contribution of the higher modes to the system output is less than the lower modes. It has the form

$$J_1 = -\sum_{r=1}^{m}\sum_{i=1}^{k} B_{ri}^2 - \sum_{r=m+1}^{n}\sum_{i=1}^{k}\frac{B_{ri}^2}{m+1} \tag{33}$$

The minus sign is present because the optimization method used is a minimization procedure. The second objective function maximizes the elements of the B matrix corresponding to the controlled modes (B_M) and minimizes the elements associated with the residual modes. This objective function insures that each targeted mode is controlled. It is selected as

$$J_2 = \sum_{r=1}^{m}\frac{1}{\sum_{i=1}^{k}|B_{ri}|} + \sum_{r=m+1}^{n}\frac{\sum_{i=1}^{k}|B_{ri}|}{m+1} \tag{34}$$

These objective functions can be used in conjunction with any control law. We next develop an objective function considering decentralized collocated control. From Eq. (32) we note that the coupling among the modes is given by the BKB^T matrix. The objective function is selected to minimize the energy transfer from one mode onto another by diagonalizing BKB^T and has the form

$$J_3 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(BKB^T)_{ij}(1-\delta_{ij})}{(BKB^T)_{ii}} \tag{35}$$

We now develop an objective function considering modal control. For illustrative purposes, we consider the independent modal-space control method (IMSC).[19] From Eq. (23a), the relation between the modal forces and actual forces (or torques) is $f_M(t) = B_M F(t)$, which, when inverted, yields the actual control inputs as $F(t) = B_M^{-1} f_M(t)$.

The modal work terms $W_r(r = 1, 2, \ldots, m)$ corresponding to the controlled modes do not change when IMSC is applied. However, as in other modal control laws, the modal work for the residual modes is no longer zero, $W_r \neq 0$ $(r = m+1, m+2, \ldots)$, indicating wasted energy. The expression for the

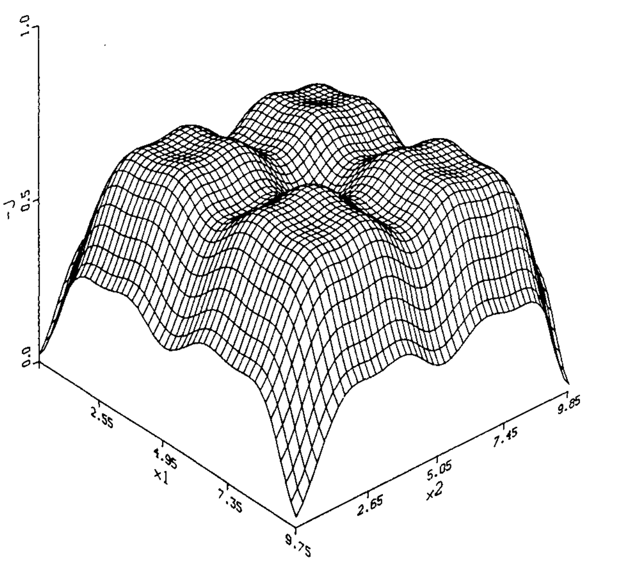

Fig. 1 Plot of J_1 for force actuators.

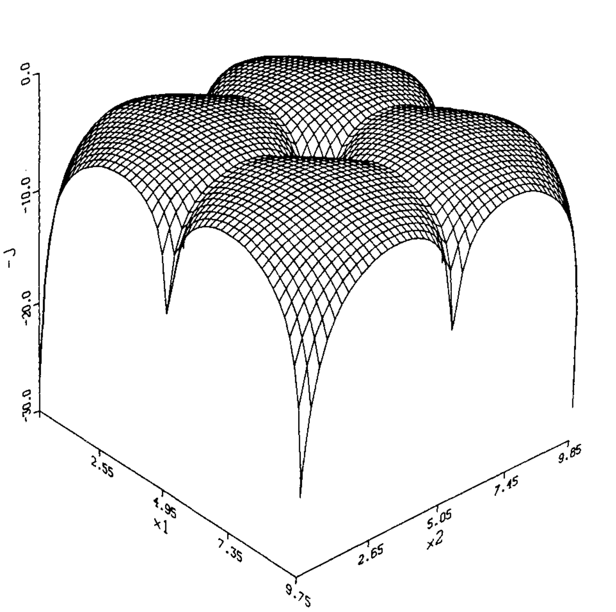

Fig. 2 Plot of J_2 for force actuators.

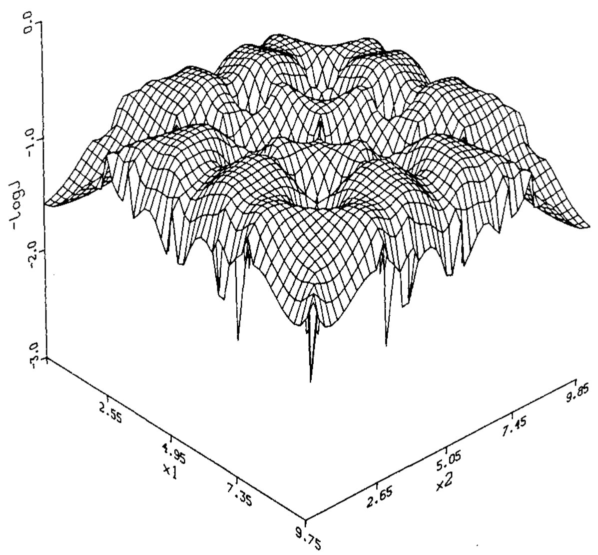

Fig. 3 Plot of J_3 for force actuators.

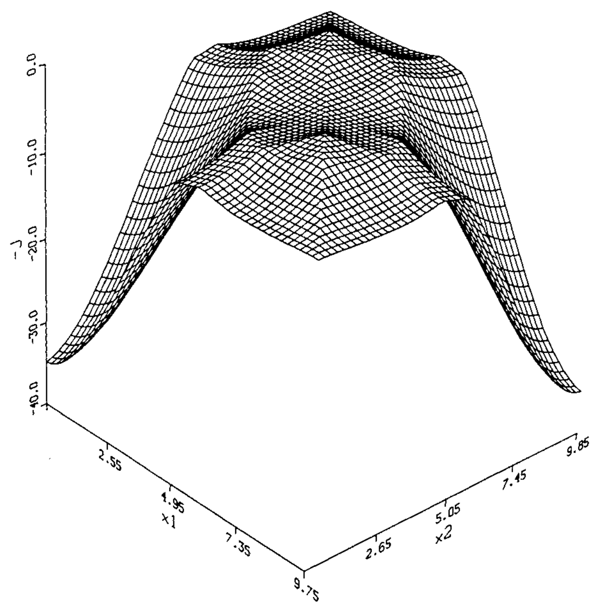

Fig. 4 Plot of J_4 for force actuators.

Table 1 Work and fuel for IMSC using force actuators (initial conditions: $\dot{u}(0) = [1, -1, -1, 1]^T$)

Minimizing	(x_1, x_2)	$H(20)$	W	W_A	W_M	$\mathfrak{F}$	$W/\mathfrak{F}$
J_1	3.45, 6.55	1.042	−0.962	1.283	1.779	3.498	0.275
J_2	3.05, 6.95	1.077	−0.925	1.008	1.599	3.399	0.272
J_3	1.95, 4.05	1.585	−0.417	2.042	5.093	8.253	0.050
J_4	2.55, 7.45	1.151	−0.849	1.093	1.580	3.580	0.237

Table 2 Modal energy at $t = 20$ s: IMSC with force actuators ($\dot{u}(0) = [1, -1, -1, 1]^T$)

(x_1, x_2)	H	H_1	H_2	H_3	H_4
3.45, 6.55	1.0423	0.0455	0.0093	0.4910	0.4964
3.05, 6.95	1.0777	0.0455	0.0093	0.5277	0.4951
1.95, 4.05	1.5852	0.0455	0.0093	0.8123	0.7180
2.55, 7.45	1.1516	0.0455	0.0093	0.5976	0.4991
at $t = 0$	2.0000	0.5000	0.5000	0.5000	0.5000

Table 3 Work and fuel for collocation using force actuators (initial conditions: $\dot{u}(0) = [1, -1, -1, 1]^T$)

(x_1, x_2)	$H(20)$	W	W_A	W_M	$\mathfrak{F}$	$W/\mathfrak{F}$
3.45, 6.55	0.985	−1.016	1.017	1.115	2.259	0.450
3.05, 6.95	1.075	−0.927	0.927	1.044	2.200	0.420
1.95, 4.05	0.769	−1.231	1.231	1.413	2.636	0.467
2.55, 7.45	1.135	−0.865	0.865	0.951	2.227	0.388

Table 4 Modal energy at $t = 20$ s: collocation with force actuators ($\dot{u}(0) = [1, -1, -1, 1]^T$)

(x_1, x_2)	H	H_1	H_2	H_3	H_4
3.45, 6.55	0.9851	0.1972	0.1725	0.4878	0.1275
3.05, 6.95	1.0748	0.2302	0.1263	0.4553	0.2628
1.95, 4.05	0.7695	0.2110	0.1769	0.1921	0.1892
2.55, 7.45	1.1353	0.2773	0.1034	0.2574	0.4970
at $t = 0$	2.0000	0.5000	0.5000	0.5000	0.5000

modal forces associated with the residual modes becomes

$$f_R(t) = B_R F(t) = B_R B_M^{-1} f_M(t) \tag{36}$$

One objective in modal control is to minimize the excitation going into the residual modes. We then select the following objective function:

$$J_4 = \sum_{r=1}^{n-m} \sum_{i=1}^{m} |(B_R B_M^{-1})_{ri}| \tag{37}$$

It should be noted that one can propose several other objective functions based on other criteria. The objective functions are not functions of time, so that they do not give an indication as to how the work and fuel consumption will be for a given set of actuator locations. To get quantitative measures after the optimal locations are selected, one should calculate and compare the work, fuel, and final energy for the different actuator locations. We carry out this analysis in the next section.

VI. Optimal Actuator Placement

In this section, we suppress the vibration of a pinned-pinned beam of length 10 and with unit mass and stiffness distributions using point force actuators, point torque actuators, and piecewise-continuous force actuators.

Point Force Actuators

We first plot in Figs. 1-4 the objective functions using two actuators and four modes in the mathematical model. From these figures we observe the following.

1) Actuator locations near the boundaries yield larger values for the objective functions, which is to be expected.

2) In all plots, there exist several local minima.

3) For J_1 and J_2, the objective functions are symmetric about the $x_1 = x_2$ line.

4) The objective function J_3 is very sensitive to the variation in the actuator locations. Also, the amplitudes of the objective function are higher than the other objective functions. Note that the z axis is labeled $-\log J_3$.

5) From the plots of J_3 and J_4, it is observed that the two actuators should not be very close to each other. When $x_1 = x_2$ we essentially have a single actuator. The objective functions J_1 and J_2 do not penalize closely spaced actuators.

The actuator locations that minimize each of the objective functions are used to compare the work, fuel, and energy in Tables 1-4. We consider two control laws: collocation as a decentralized control law, and independent modal-space control as a modal control law. The control gains are selected as $f_r(t) = -0.2\,\dot{u}_r(t)$ $(r = 1, 2, \ldots, n)$ for IMSC and $F_j(t) = -0.2\,\dot{u}(x_j, t)$ $(j = 1, 2, \ldots, k)$ for collocation.

The initial conditions are chosen such that the modal velocities have magnitudes of the same order: $\dot{q}(t) = [1.0 \quad -1.0 \quad -1.0 \quad 1.0]^T$. The initial displacements are zero. The control is carried out for 20 s. Table 1 gives the work and fuel used for the IMSC method. Table 2 lists the energy levels. The second set of locations leads to the best fuel efficiency and largest amount of work done on the system. The third set gives the worst results. The locations that minimize J_1, J_2, and J_4 give comparable results. These actuator locations are the most evenly distributed.

Tables 3 and 4 display the work, fuel, and energy levels at the end of the control for the collocation control law. We observe that the third set of locations gives the best work performance. Note that the third objective function is designed considering collocation. The control performance for the different actuator locations is not as different as when IMSC is used, indicating that collocation is less sensitive to the actuators locations.

A similar analysis was performed by considering initial conditions where the higher modes have lower amplitudes. The effects of spillover were found to be less drastic. For both types of control, there was less difference in all of the performance measures for the different locations.

We increase the number of actuators to five and optimize the actuators' locations. The golden section search method combined with a first-order gradient technique[20] is used to find the minimum values of the objective functions. We include 10 modes in the model with the lowest five being controlled by five actuators. With this larger number of actuators considered, each objective function has several local minima, and it becomes difficult to determine whether the global minimum is reached. From the results of plotting the objective functions earlier, we observe that a relatively even distribution of the actuators gives comparable values of the objective function. To investigate the general behavior of each objective function, we choose four initial actuator location sets as i) [1.66, 3.33. 5.00, 6.67, 8.33], ii) [1.00, 3.00, 5.00, 7.00, 9.00], iii) [3.00, 4.00, 5.00, 6.00, 7.00], and iv) [2.00, 3.00, 4.00, 5.00, 6.00] and carry out the optimization. The corresponding local minima are given in Table 5. As in the case of two actuators, when the actuator locations are relatively evenly distributed, magnitudes of the objective functions are

Table 5 Local minima for five force actuators using different objective functions

	x_1	x_2	x_3	x_4	x_5	J_1	J_2	J_3	J_4
				J_1					
i[a]	1.209	3.179	5.000	6.820	8.785	−3.337	—	—	—
ii	1.215	3.185	5.000	6.815	8.785	−3.337	—	—	—
iii	3.185	4.006	5.000	5.994	6.815	−3.058	—	—	—
iv	1.212	3.183	4.987	5.000	5.013	−3.252	—	—	—
				J_2					
i	1.429	3.222	5.000	6.584	8.571	—	3.629	—	—
ii[a]	1.429	2.954	5.000	7.046	8.571	—	3.597	—	—
iii	2.859	3.750	5.000	6.250	7.141	—	3.953	—	—
iv	1.386	2.857	3.711	5.000	6.289	—	3.756	—	—
				J_3					
i[a]	1.072	2.177	3.308	4.429	7.750	—	—	4.969	—
ii	1.000	3.000	5.000	7.000	9.000	—	—	6.000	—
iii	2.851	3.996	5.000	6.004	7.149	—	—	6.701	—
iv	1.686	2.833	4.014	5.211	6.392	—	—	5.114	—
				J_4					
i[a]	1.243	3.752	4.992	6.253	8.743	—	—	—	2.611
ii	1.111	2.970	5.014	7.031	8.952	—	—	—	3.751
iii	1.293	3.732	5.044	7.470	8.789	—	—	—	2.696
iv	1.111	2.261	3.883	5.153	6.617	—	—	—	6.950

[a]Minimum J among actuator location sets.

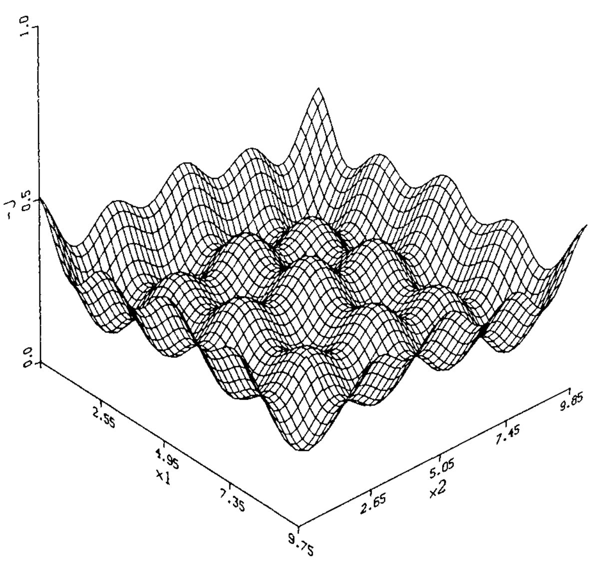

Fig. 5 Plot of J_1 for torque actuators.

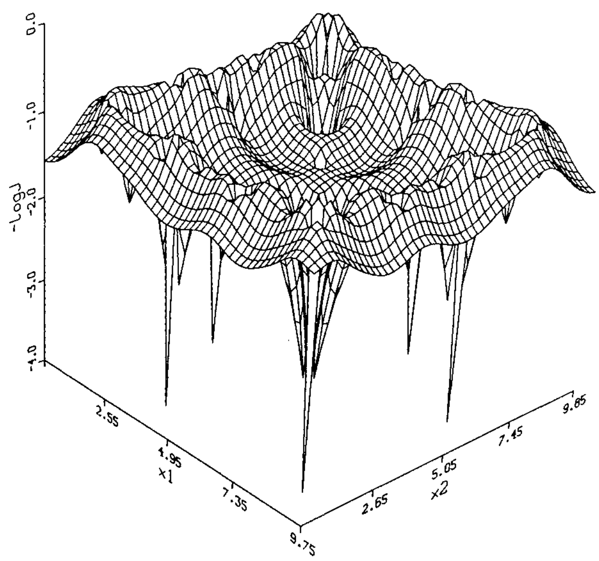

Fig. 7 Plot of J_3 for torque actuators.

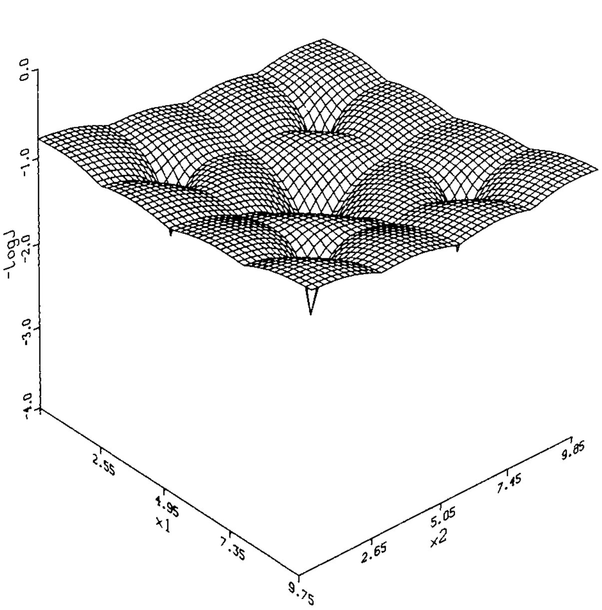

Fig. 6 Plot of J_2 for torque actuators.

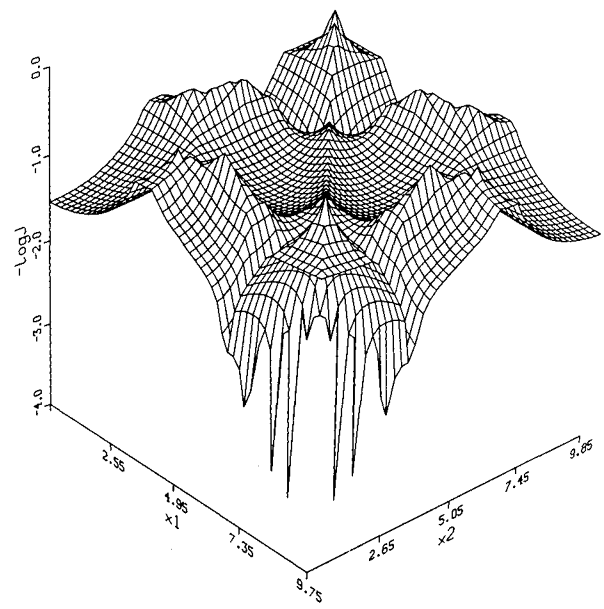

Fig. 8 Plot of J_4 for torque actuators.

Table 6 Work and fuel for IMSC using force actuators (initial conditions: $\dot{u}(0) = [1, -1, -0.3, 0.3]^T$)

Minimizing	(x_1, x_2)	$H(20)$	W	W_A	W_M	$\mathfrak{F}$	$W/\mathfrak{F}$
J_1, J_2	0.15, 9.85	0.518	−0.571	1.641	2.940	7.355	0.078
J_3	1.55, 8.65	0.188	−0.903	0.983	1.275	9.317	0.097
J_4	1.25, 8.75	0.219	−0.873	1.025	1.281	8.564	0.102

Table 7 Modal energy at $t = 20$ s: IMSC with torque actuators ($\dot{u}(0) = [1, -1, -0.3, 0.3]^T$)

(x_1, x_2)	H	H_1	H_2	H_3	H_4
0.15, 9.85	0.5181	0.0456	0.0093	0.3752	0.0880
1.55, 9.65	0.1882	0.0456	0.0093	0.0890	0.0444
1.25, 8.75	0.2191	0.0456	0.0093	0.1193	0.0450
at $t = 0$	1.0900	0.5000	0.5000	0.0450	0.0450

Table 8 Work and fuel for collocation using force actuators (initial conditions: $\dot{u}(0) = [1, -1, -0.3, 0.3]^T$)

(x_1, x_2)	$H(20)$	W	W_A	W_M	$\mathfrak{F}$	$W/\mathfrak{F}$
0.15, 9.85	0.727	−0.363	0.363	0.419	1.460	0.249
1.55, 8.65	0.928	−0.162	0.162	0.183	0.968	0.167
1.25, 8.75	0.897	−0.193	0.193	0.203	1.088	0.178

Table 9 Modal energy at $t = 20$ s: collocation with torque actuators ($\dot{u}(0) = [1, -1, -0.3, 0.3]^T$)

(x_1, x_2)	H	H_1	H_2	H_3	H_4
0.15, 9.85	0.7272	0.4423	0.2687	0.0129	0.0033
1.55, 8.65	0.9280	0.4524	0.3958	0.0424	0.0374
1.25, 8.75	0.8974	0.4494	0.3654	0.0376	0.0450
at $t = 0$	1.0900	0.5000	0.5000	0.0450	0.0450

comparable. Also, the objective functions give similar optimal actuator locations.

We conclude that, as the number of actuators is increased, their placement becomes less of a problem, a result that is intuitive as well. For systems of high order and having a large number of actuators the optimization procedure will become even more complicated, with several more local minima. We propose to treat such cases by quantifying the number of possible actuator locations and using a search procedure that selects the optimal distribution from the set of possible locations.

Point Torque Actuators

We now conduct for point torque actuators the same analysis as force actuators. Based on the results of Ref. 15, we expect torque actuators to fare more poorly.

We first plot in Figs. 5–8 the four objective functions when two torque actuators are used with four modes in the mathematical model. The plots, when compared with Figs. 1–4, look like their inverses. They have many more local minima, and actuator locations near the boundaries yield smaller values for the objective functions. The amplitudes of the objective functions have larger variations than for force actuators, an indication that the objective functions are more sensitive to the variation of actuator locations. As in the force actuator case, having two actuators close to each other is not desirable. These observations, together with the results of Ref. 15, indicate that placement of torque actuators is more critical than placement of force actuators.

As in the point force actuator case, the actuator location pairs that minimize each of the objective functions are used to compare the control performance. The first and second objective functions yield the same torquer locations. We consider the same control laws and control gains as in the force actuator example. The initial conditions are selected as $\dot{u}(t) = [1.0\ -1.0\ -0.3\ 0.3]^T$ with no initial displacements. Tables 6–9 display the counterparts of Tables 1–4 for torque actuators. From these tables, we observe that all indicators of performance, such as the work done, fuel used, and energy left in system, are higher when torque actuators are used. In addition, there is substantial difference in performance for the three different actuator locations. Again, the locations corresponding to J_1, J_2, and J_4 give the best results.

We increase the number of actuators to five, and find the minimum values of the objective functions for a 10-mode model. We begin with five sets of initial actuator distributions. The first four sets are the same as the force actuator case. In the fifth set, we try an initial actuator distribution that places actuators near the boundaries, v) [0.1, 2.5, 5, 7.5, 9.9]. The motivation behind selecting this particular set is the observation from Figs. 5–8 that the values of J_i have lower values near the boundaries.

The results, given in Table 10, show improvement over the two actuator case. The optimal actuator locations are relatively evenly distributed for all of the objective functions. In most cases actuators near the boundaries give better results. There is a large variation in the optimal locations for J_3 and J_4 depending on the initial actuator distributions. Also, the optimization does not converge for some of the initial locations. We conclude that for torque actuators similar to the force actuator case, placement becomes less critical as the number of actuators is increased. The actuator placement is more difficult than for force actuators. Also, as shown earlier, torquers excite the higher modes more and use more energy and more fuel and leave the system with more internal energy at the end of the control cycle. These disadvantages should be compared with the advantages of using torque actuators, such as their being energy storage devices and that they do not saturate easily.

Piecewise-Continuous Force Actuators

Actuators in general have finite contact areas. Also, there are future plans to build actuators with large surface areas. The advantage of a piecewise-continuous actuator is that it reduces the stress levels and it excites the higher modes less. Here, we compare the performance of piecewise-continuous force actuators as a function of the actuator length.

We consider the two control laws and the four-mode model with two force actuators used in the previous examples. Tables 11 and 12 compare the work and fuel for different actuator lengths. Note that $\epsilon = 1$ corresponds to an actuator of length 20% of the length of the beam.

From Tables 11 and 12, we observe that increasing the contact length of the actuators results in more fuel use and decreases the work fuel ratio. For IMSC the actual work increases and there is less energy left in the system. For collocation, the actual work is less and there is more energy left in the system. However, for both types of control, the variations are small. The increase in actual work is due to the force density of the actuator being the same throughout the contact length.[15] If there is a sign change in the velocity of the beam within the contact length, the actuator performs both positive and negative work.

When the contact length is small, Tables 11 and 12 indicate that the difference in performance is almost negligible, so that piecewise-continuous actuators present desirable alternatives to point actuators. On the other hand, increasing the contact length too much reduces the efficiency of the control. One

Table 10 Local minima for five torque actuators using different objective functions

	x_1	x_2	x_3	x_4	x_5	J_1	J_2	J_3	J_4
				J_1					
i	2.402	3.923	5.000	6.047	7.606	−4.562	—	—	—
ii	2.403	2.404	5.000	7.597	7.597	−4.583	—	—	—
iii	2.404	3.930	5.000	6.069	7.596	−4.562	—	—	—
iv	2.402	2.404	3.930	5.000	6.070	−4.563	—	—	—
v[a]	0.006	2.403	5.000	7.597	9.994	−5.805	—	—	—
				J_2					
i	1.877	3.292	5.000	6.709	7.858	—	7.351	—	—
ii	0.625	3.559	5.000	6.441	9.375	—	6.286	—	—
iii	3.125	3.552	5.000	6.448	6.875	—	8.187	—	—
iv	2,030	3.125	3.950	5.000	6.050	—	7.964	—	—
v[a]	0.625	2.143	5.000	7.857	9.375	—	5.833	—	—
				J_3					
i	2.007	3.3666	4.996	6.637	7.993	—	—	23.714	—
ii				No convergence					
iii	2.267	3.821	5.000	6.179	7.732	—	—	26.921	—
iv[a]	0.729	2.095	3.609	4.854	6.592	—	—	11.220	—
v	0.547	1.864	5.000	8.136	9.453	—	—	12.715	—
				J_4					
i	2.664	4.445	6.752	7.463	9.227	—	—	—	8.603
ii[a]	0.670	1.882	3.304	5.133	9.745	—	—	—	6.123
iii	0.747	2.255	4.215	5.994	6.808	—	—	—	7.356
iv	2.168	3.143	4.193	5.409	6.812	—	—	—	17.750
v				No convergence					

[a]Minimum J among actuator location sets.

Table 11 Work and fuel for different actuators lengths [IMSC, $(x_1, x_2) = (3.45, 6.55)$]

ϵ	$H(20)$	W	W_A	W_M	$\mathfrak{F}$	$W/\mathfrak{F}$
0.00	0.1494	−0.9434	0.9634	1.1978	3.4989	0.269
0.01	0.1494	−0.9434	0.9638	1.1977	3.4987	0.269
0.10	0.1494	−0.9434	0.9644	1.1989	3.5003	0.269
0.50	0.1486	−0.9440	0.9712	1.1859	3.5417	0.266
1.00	0.1466	−0.9453	0.9842	1.1477	3.6788	0.256

Table 12 Work and fuel for different actuators lengths [collocation, $(x_1, x_2) = (3.45, 6.55)$]

ϵ	$H(20)$	W	W_A	W_M	$\mathfrak{F}$	$W/\mathfrak{F}$
0.00	0.9852	−1.0168	1.0168	1.1151	2.2596	0.450
0.01	0.9852	−1.0168	1.0168	1.1151	2.2596	0.450
0.10	0.9858	−1.0162	1.0162	1.1164	2.2605	0.450
0.50	1.0010	−1.0003	1.0084	1.0999	2.2821	0.438
1.00	1.0520	−0.9475	0.9963	1.0463	2.3488	0.403

should compare the reduction in stress levels and the deterioration in the control performance when deciding on the contact length to use.

VII. Conclusions

The placement of force and torque actuators in structural control problems is considered. Objective functions are defined based on the elements of the actuator influence matrix, and optimization studies are conducted. The performance of the control is compared for fuel use and work done. The studies indicate that a relatively even distribution of the actuators gives satisfactory results, whereas a very close spacing of the actuators leads to excessive fuel and energy use. As the number of actuators is increased, the actuators become less sensitive to their placement. The performance of torque actuators is more sensitive to their placement than force actuators. Torque actuators are also less efficient because they excite the residual dynamics more and use more energy and fuel. Designing actuators as piecewise continuous reduces stress levels. However, if the contact length is too large, such actuators become less efficient and use more fuel than point actuators. It is recommended that one use several objective functions before deciding on which actuator locations to use.

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